Nafta-Gaz 2023, no. 11, pp. 736–741, DOI: 10.18668/NG.2023.11.06

Derivation of phenomenological equations of hydromechanics of multi-phase flows

Wyprowadzenie fenomenologicznych równań hydromechaniki przepływów wielofazowych

Gasim A. Mamedov, Rauf Kh. Malikov, Natiq M. Abbasov, Mahluga S. Rahimova

Azerbaijan State Oil and Industry University

ABSTRACT: In the article, a multi-phase (non-homogeneous, heterogeneous) medium is considered as a macrosystem (continuum) composed of several (at least two) phases, such as a carrier phase (liquid, vapor or gas) and a carried phase (solid particles, bubbles or drops).The masses and mixtures of these phases undergo continuous changes over time due to the addition or separation of new masses to or from both phases. The model takes into account interphase transitions, discontinuities inside the mixture, and the possibility of phases being either continuous or discrete, depending on their location. A method for preliminary smoothing of discontinuities has been developed, leveraging the fact that the location in space, as well as the shape and size of the discrete phase are random. A function, denoted as $\varphi_i(x, y, z, t)$, has been introduced, which indicates the probability of the presence of the i-th phase in the vicinity of a given point in space at time t, or that the given point of space x, y, z at time t belongs to the set of points of the i-th phase. On the other hand, this probability can be interpreted as the volumetric concentration of the i-th phase at a given point in space (i.e., the ratio of the measure of the set of points belonging to the i-th phase in the vicinity of the point under consideration at time t to the measure of the entire set of points in the surrounding area). This hypothetical medium, being equivalent to the original one, serves as a model for a multi-phase (inhomogeneous, heterogeneous, two-phase) medium. The uniqueness of the model arises from its construction. In addition, this paper considers several main areas of theoretical and experimental research concerning the hydrodynamics of a multi-phase (two-phase suspension-carrying) flow of a continuous medium. It also discusses the most important results achieved in existing works. A critical analysis of known theories for mathematically describing the motion of multi-phase (two-phase) systems and methods for averaging the hydrodynamic characteristics of a turbulent flow are given. The procedure for closing the equations systems of hydromechanics of multi-phase flows proposed in existing works is carried out within the framework of semi-empirical theories of turbulence. In nature, the vast majority of multi-phase (two-phase, inhomogeneous) mixtures exhibit turbulent behavior, making its study a crucial practical task. The mathematical description of the motion of a turbulent multi-phase flow relies on stylized laws of mechanics. The methods of operational analysis proposed at various times by different researchers for the mathematical description of the motion of a multi-phase (two-phase) flow have varying degrees of approximation and certain limited areas of application. One of the main challenges in formulating differential equations for the motion of a turbulent multi-phase (two-phase, suspension-carrying) flow is the fact that in a turbulent flow of a mixture, where the characteristics of the flow change chaotically and randomly over time and at each point in space, both in magnitude and in direction, there are surfaces with weak and strong discontinuities. Consequently, the actual values of velocity and pressure of a multi-phase flow, strictly speaking, cannot be considered continuous functions of the coordinates of space and time throughout the entire region occupied by the mixture.

Key words: hydromechanics of multi-phase flows; mass transfer equations; momentum equations; kinetic energy equations; total energy equations and multi-phase medium.

STRESZCZENIE: Niniejszy artykuł omawia medium wielofazowe (niejednorodne, heterogeniczne), jako makrosystem (kontinuum) składający się z kilku (co najmniej dwóch) faz, takich jak faza nośna (ciecz, para lub gaz) i faza niesiona (cząstki stałe, pęcherzyki lub krople). Masy i mieszaniny tych faz ulegają ciągłym zmianom w czasie z powodu dodawania lub oddzielania nowych mas do lub z obu faz. Model uwzględnia przejścia międzyfazowe, nieciągłości wewnątrz mieszaniny oraz możliwość występowania faz ciągłych lub rozproszonych, w zależności od ich położenia. Opracowano metodę wstępnego wygładzania nieciągłości, wykorzystując fakt, że lokalizacja w przestrzeni, a także kształt i rozmiar fazy rozproszonej są losowe. W modelu tym wprowadzono funkcję wyrażającą prawdopodobieństwo obecności i-tej fazy w pobliżu danego punktu przestrzeni w czasie t lub tego, że dany punkt przestrzeni w czasie t należy do zbioru punktów i-tej fazy. Z drugiej strony, prawdopodobieństwo to można interpretować jako stężenie objętościowe i-tej fazy w danym punkcie przestrzeni (tj. stosunek miary zbioru punktów należących do i-tej fazy w sąsiedztwie rozpatrywanego punktu

Corresponding author: M.S. Rahimova, e-mail: *rahimova_mahluqa@mail.ru*

Article contributed to the Editor: 23.03.2023. Approved for publication: 18.10.2023.

w chwili t do miary całego zbioru punktów w otoczeniu). Ten hipotetyczny ośrodek, będąc równoważnym oryginalnemu, służy jako model ośrodka wielofazowego (niejednorodnego, heterogenicznego, dwufazowego). Wyjątkowość modelu wynika z jego konstrukcji. Ponadto, w artykule omówiono kilka głównych obszarów badań teoretycznych i eksperymentalnych dotyczących hydrodynamiki wielofazowego (dwufazowego) przepływu zawiesiny w medium ciągłym. Omówiono również najważniejsze wyniki uzyskane w istniejących pracach. Dokonano krytycznej analizy znanych teorii matematycznego opisu ruchu układów wielofazowych (dwufazowych) oraz metod uśredniania charakterystyk hydrodynamicznych przepływu turbulentnego. Zaproponowana w istniejących pracach procedura rozwiązywania układów równań hydromechaniki przepływów wielofazowych jest realizowana w ramach półempirycznych teorii turbulencji. W naturze zdecydowana większość mieszanin wielofazowych (dwufazowych, niejednorodnych) wykazuje zachowanie turbulentne, co czyni ich badanie kluczowym zadaniem praktycznym. Matematyczny opis ruchu turbulentnego przepływu wielofazowego opiera się na uproszczonych prawach mechaniki. Metody analizy operacyjnej zaproponowane w różnym czasie przez różnych badaczy do matematycznego opisu ruchu przepływu wielofazowego (dwufazowego) charakteryzują się różnym stopniem przybliżenia i pewnymi ograniczonymi obszarami zastosowań. Jednym z głównych wyzwań w formułowaniu równań różniczkowych dla ruchu turbulentnego przepływu wielofazowego (dwufazowego, przenoszącego zawiesinę) jest fakt, że w turbulentnym przepływie mieszaniny, gdzie charakterystyka przepływu zmienia się chaotycznie i losowo w czasie oraz w każdym punkcie przestrzeni, zarówno pod względem wielkości, jak i kierunku, występują powierzchnie o słabych i silnych nieciągłościach. W związku z tym rzeczywiste wartości prędkości i ciśnienia przepływu wielofazowego, ściśle rzecz biorąc, nie mogą być uważane za ciągłe funkcje współrzędnych przestrzeni i czasu w całym obszarze zajmowanym przez mieszaninę.

Słowa kluczowe: hydromechanika przepływów wielofazowych; równania przepływu masy; równania pędu; równania energii kinetycznej; równania energii całkowitej i medium wielofazowego.

Task setting

It is assumed that there are interphase transitions, discontinuities within the mixture, and that the phases, depending on their location, can be either continuous or discrete. A method for preliminary smoothing of discontinuities has been developed, leveraging the fact that the location in space, as well as the shape and size of the discrete phase are random. A function, denoted as $\varphi_i(x, y, z, t)$ has been introduced, which indicates the probability of the presence of the i-th phase in the vicinity of a given point of space x, y, z at time t, or that a given point of space x, y, z at that moment of time belongs to the set of points of the i-th phase.

Goal of the work

The goal of this study is to derive the general equations of hydromechanics for multi-phase flows, consisting of the equations of mass transfer, momentum, angular momentum, kinetic and total energy.

Introduction

The range of problems related to the hydromechanics of multi-phase (two-phase suspended, heterogeneous, inhomogeneous) media is remarkably extensive and has experienced significant development in recent years. This growth is driven by its critical practical applications in areas such as: oil and gas wells drilling; hydro-, thermal and nuclear power engineering; aviation and rocket technology; stratification and ecology; hydraulic engineering and water management; pipeline transport of oil, gas, water and other liquids; petrochemistry; chemical technology and many others. A characteristic feature of multi-phase media is the coexistence of carrier (liquid) and carried (suspended) phases (gas-solid particles, gas-liquid drops, liquid-gas bubbles, liquid-solid particles, vapor-liquid drops, etc.). In such flows, there is a constant exchange of mass, momentum and energy (kinetic and thermal) between these phases. In addition, a specific feature of the system under consideration is also the fact that, when both phases can be regarded as incompressible, the multi-phase medium exhibits behavior akin to that of a compressible fluid.

Experimental part

To derive the relevant equations of hydromechanics, an arbitrary volume of a multicomponent medium limited by the surface is singled out. For the sake of generality, it is assumed that the total mass of the mixture undergoes continuous changes over time due to the addition to it (or separation from it) of the elementary mass (Figure 1).

To mathematically describe the motion of a multicomponent flow, specific characteristics are employed. Let an elementary volume τ with mass m contain i phases with volumes $\tau_1, \tau_2, \tau_3, ..., \tau_i$ and masses $m_1, m_2, m_3, ..., m_i$.

Then:

$$\phi_i = \frac{\tau_i}{\tau} \quad (1)$$

- volume concentration (share) of the i-th phase

$$\rho_i = \frac{m_i}{\tau_i} \quad (2)$$

- *i*-th phase density

$$\rho = \frac{m}{\tau} \tag{3}$$

- medium density

$$x_i = \frac{m_i}{m} = \frac{\phi_i \rho_i}{\rho} \tag{4}$$

mass concentration of *i*-th phase.

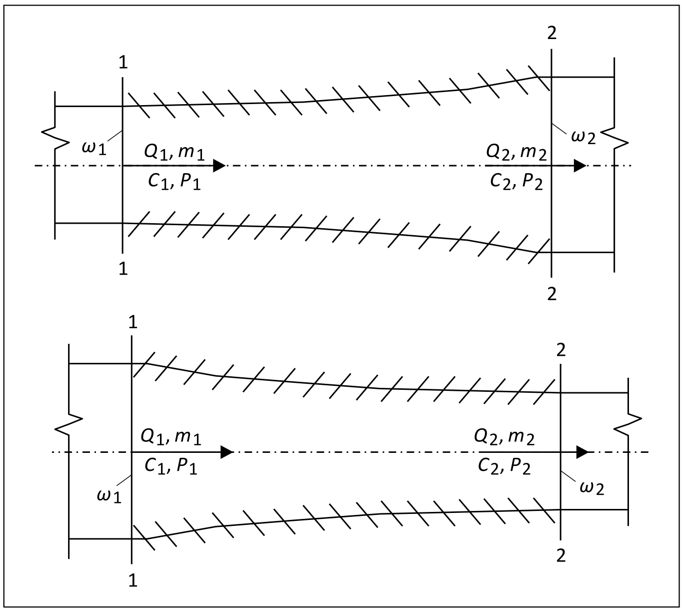

Figure 1. Flow diagram with mass attachment or separation

Rysunek 1. Schemat przepływu z przyłączeniem masy lub jej oddzieleniem

It is evident that:

$$\tau = \sum_i \tau_i,\ m = \sum_i m_i,\ \rho = \sum_i \rho_i \phi_i,\ \sum_i \phi_i = 1,\ \sum_i x_i = 1 \tag{5}$$

Therefore, the geometric characteristic of the volume τ at each point in the space is φ_i. In this case, it is possible to determine the parameters characterizing the environment as a whole, namely:

- mixture velocity vector

$$\bar{C} = \frac{1}{\rho} \sum_i \rho_1 \phi_i \bar{C}_i = \sum_i x_1 \bar{C}_i \tag{6}$$

- mixture mass force vector

$$\bar{F} = \frac{1}{\rho} \sum_i \rho_1 \phi_i \bar{F}_i \tag{7}$$

- mixture surface force tensor

$$\Pi = \sum \phi_i \Pi_i \tag{8}$$

and so on. Here $\bar{F}_i$, Π_i are external mass and surface stresses acting on the *i*-th phase of the mixture.

The derivation of the general equations of mass transfer, momentum, angular momentum, kinetic and total energy in two-phase media with a change in mass flow is described below. Two different construction methods are employed for this purpose. The first method involves forming conservation laws (for mass, momentum and energy) for each component of the mixture (Zvonarev, 2019). The second approach involves considering that the phases are distributed within one another. In this case, either one of the phases or both phases are treated as continuous throughout the entire considered volume of the mixture, and the equations characterizing the course of the process in them are written for the mixture as a whole. This approach is typically used to describe so-called homogeneous mixtures (media) consisting of well-dispersed components within a medium (liquid or gaseous), as well as solutions, i.e., in cases where the interaction between the components of the mixture actually occurs at the molecular level.

Here, in a general form (in a three-dimensional form), the construction of systems of equations of hydromechanics that describe the motion of each phase separately and the medium as a whole is considered (Alder, 2001). To establish these equations, an arbitrary volume of a multi-phase medium $\tau(t)$ limited by the surface $\sigma(t)$ is selected, and it is assumed that the total mass of the mixture changes continuously (Shen, 2012).

As it is well-known, the mass transfer equation serves a mathematical formulation of the law of mass conservation. In the presence of sources of mass flow, we formulate this law as follows: the time derivative of the mass of the mixture (phase, class) within an arbitrary volume $\tau(t)$ is equal to its change resulting from the attachment to it (or separation from it) of the elementary mass and phase transformations.

1. Let us consider an arbitrary volume $\tau(t)$ bounded by surface $\sigma(t)$ and the presence of s carrier (continuous) phase within it at time *t*. Let the elementary volume $d\tau$, contain the mass of this phase $\rho_f \varphi_f d\tau$ where $\rho_f \varphi_f$ represent the density and volumetric concentration (share or content) of the carrier phase, respectively. Consequently, the total mass of the considered phase within the volume $\tau(t)$ will be expressed by the volume (triple) integral (Mathematical modeling, 2014):

$$m_f = \int_{\tau(t)} \rho_f \phi_f d\tau \tag{9}$$

where:

m_f – is the mass of the carrier phase.

The change in the mass of the carrier phase (the derivative of the mass, m_f, with respect to time, t) within the allocated volume per unit time, is expressed as:

$$\frac{dm_f}{dt} = \frac{d}{dt} \int_{\tau(t)} \rho_f \phi_f d\tau \tag{10}$$

This mass change occurs due to the process of attachment to it (or separation from it) of the elementary mass of the carrier phase and phase transformations (for example, the transition of the carrier phase to the carried phase), per unit of time:

$$\dot{m}_f = \int_{\tau(t)} (q_f - \chi)\, d\tau \qquad (11)$$

where: q_f – the specific attached (or detached, then $q_f < 0$) mass of the carrier phase (i.e. per unit volume, the second mass attached to or detached from the carrier phase through the surface limiting the selected volume); χ – the specific mass of the phase transition of the carrier phase into the carried phase (i.e., per unit volume, the second mass of the phase transition at the interface) (Kuznetsov, 2017).

Then, in accordance with the law of conservation of mass:

$$\frac{d}{dt}\int_{\tau(t)} \rho_f \phi_f \, d\tau = \int_{\tau(t)} (q_f - \chi)\, d\tau \qquad (12)$$

Let us note that in the region of continuous motions, the integral equation of the law of conservation of mass for the carrier phase (12) will be satisfied for any moving volume $\tau(t)$ with a smooth boundary $\sigma(t)$ and is equivalent to the differential equation, to the derivation of which we proceed (Umnov, 2012).

The formula for differentiation with respect to time from the integral taken over the moving volume can be used to transform the left side of (12). Let a continuous differentiable function be defined at each point of the moving mixture (phase, class), depending on the coordinates of points in space and time (it can be a scalar, vector or tensor), i.e. $a(x, y, z, t)$. And let $J(t)$ be the function defined by the integral:

$$j(t) = \int_{\tau(t)} a(x,y,z,t)\, d\tau \qquad (13)$$

taken over the moving volume $\tau(t)$. The total time derivative of $J(t)$ has the form:

$$\frac{d}{dt}\int_{\tau(t)} a\, d\tau = \int_{\tau(t)} \frac{\partial a}{\partial t} d\tau + \int_{\sigma(t)} a C_n \, d\sigma \qquad (14)$$

Equation (14) is a formula for differentiating with respect to time from an integral taken over a moving volume $\tau(t)$ with a smooth boundary (it is also called the transfer theorem). It states that the rate of change of some extensive physical quantity $a(x, y, z, t)$ (the value is considered extensive if it depends on the volume of the physical system under consideration) in the part of the medium that currently occupies the volume $\tau(t)$ is equal to the sum of changes in this quantity at all points inside the volume $\tau(t)$, plus the flow of the quantity $a(x, y, z, t)$ through the surface $\sigma(t)$ bounding the volume $\tau(t)$.

Now, in expression (14), by setting $a = \rho_f \varphi_f$, equation (12) can be represented as:

$$\int_{\tau(t)} \left[\frac{\partial}{\partial t}(\rho_f \phi_f) - (q_f - \phi) \right] d\tau + \int_{\sigma(t)} \rho_f \phi_f C_{fn}\, d\sigma = 0 \qquad (15)$$

where $C_{fn} = \overline{C}_f \cdot \overline{n}$; $\overline{C}_f$ carrier phase velocity vector, $\overline{n}$ – the outer normal to the surface.

In the last equation (15):

$\int_{\sigma(t)} \rho_f \phi_f C_{fn}\, d\sigma - \int_{\tau(t)} (q_f - \chi)\, d\tau$ represents the change in the mass of the carrier phase, contained within an arbitrarily allocated volume, per unit of time;

$\int_{\tau(t)} \frac{\partial}{\partial t}(\rho_f \phi_f)\, d\tau$ represents the rate of change in the mass of the carrier phase contained in the volume $\tau(t)$.

Equation (15) represents the mass transfer equation for the carrier phase of a multi-phase medium in integral form.

From the integral form of the mass flow transfer equation (15) for the volume, one can go over to the transfer equation at each point in space. To do this, we need to transform the surface integral (last term) in equation (15) to a volume integral. Let us express C_{fn} in terms of velocity projections (u_f, v_f, w_f) on the coordinate axes:

$$C_{fn} = \overline{C}_f \cdot \overline{n} = u_f \cos(\overline{n}, x) + v_f \cos(\overline{n}, y) + w_f (\overline{n}, z) \qquad (16)$$

and proceed according to the Gauss-Ostrogradsky formula (which states that the volume integral of the divergence of a vector over an arbitrarily chosen area is equal to the vector flow through the boundary of this area, oriented in the direction of its outer normal) to the volume integral:

$$\int_{\sigma(t)} \rho_f \phi_f C_{fn}\, d\sigma = $$
$$= \int_{\tau(t)} \left[\frac{\partial}{\partial x}(\rho_f \phi_f u_f) + \frac{\partial}{\partial y}(\rho_f \phi_f v_f) + \frac{\partial}{\partial z}(\rho_f \phi_f w_f) \right] d\tau \qquad (17)$$

or

$$\int_{\sigma(t)} \rho_f \phi_f C_{fn}\, d\sigma = \int_{\tau(t)} div(\rho_f \phi_f \overline{C}_f)\, d\tau \qquad (18)$$

Substituting (18) into (15), we get:

$$\int_{\tau(t)} \left[\frac{\partial}{\partial x}(\rho_f \phi_f) + div(\rho_f \phi_f \overline{C}_f) - (q_f - \chi) \right] d\tau = 0 \qquad (19)$$

Equation (19) holds for any volume. This is possible when the integrand equals zero. Hence:

$$\frac{\partial}{\partial x}(\rho_f \phi_f) + div(\rho_f \phi_f \overline{C}_f) = q_f - \chi \qquad (20)$$

which we will call the differential mass flow transfer equation for the carrier phase. In the absence of attached (or separated) mass of the carrier phase (i.e., at $q_f = 0$), this equation, as a special case, coincides with a similar equation for the carrier phase, derived in the works:

$$\frac{\partial}{\partial t}(\rho_f \phi_f) + div(\rho_f \phi_f \overline{C}_f) = -\chi \qquad (21)$$

2. Let us turn to the carried (discrete) phase. In the general case, the carried (discrete) phase consists of a set of particles

of varying sizes, divided into classes. First, we derive the mass flow transfer equation for a specific class of this phase. Let us consider the i-th class of the carried dispersed phase within the volume $\tau(t)$. The mass of this class in the volume $\tau(t)$ will be given by:

$$m_{si} = \int_{\tau(t)} \rho_{si}\psi_i \, d\tau \quad (22)$$

where: m_{si}, ρ_{si} mass and density of the i-th class of the carried phase,

ψ_i – distribution density of particles based on their characteristics:

$$\psi_i = \phi_s f_i \quad (23)$$

φ_s volumetric concentration (share or content) of the carried phase,

f_s distribution density of particles by classes.

Let us take into account that the mass of the i-th class of the carrier phase, as well as the carrier phase, continuously changes with time. Then, by analogy with (12), the mass conservation law in the integral form for the i-th class will be written as (Zaliznyak and Zolotov, 2023):

$$\frac{d}{dt}\int \rho_{si}\psi_i \, d\tau = \int_{\tau(t)} (q_{si} + \chi_i + \chi_{si}) d\tau \quad (24)$$

where: $\chi_{si} = \Sigma_j \chi_{sji}$ specific attached (or separated, then $q_{si} < 0$) mass of the i-th class of the carried phase; χ_{si}, χ_i – the specific masses of the phase transition of the carrier phase and the j-th class of the carried phase to the i-th class, respectively (Abbasov et al., 2006).

Using (14) and the Gauss-Ostrogradsky formulas on the left side of (24), we can obtain the following differential mass flow transfer equation for the i-th class (Allaire and Craig, 2007):

$$\frac{\partial}{\partial t}(\rho_{si}\psi_i) + div(\rho_{si}\psi_i \bar{C}_{si}) = q_{si} + \chi_i + \chi_{si} \quad (25)$$

where: $\bar{C}_{si}$ is the velocity vector of the i-th class of the carried phase. From (25) with $q_{si} = 0$ and $\chi_{si} = 0$, we get:

$$\frac{\partial}{\partial t}(\rho_{si}\psi_i) + div(\rho_{si}\psi_i \bar{C}_{si}) = \chi_i \quad (26)$$

Summing (25) over all classes i and taking into account (23), we obtain a differential mass transfer equation for the carried phase as a whole:

$$\frac{\partial}{\partial t}(\rho_s \psi_s) + div(\rho_s \psi_s \bar{C}_s) = q_s + \chi \quad (27)$$

where: ρ_s, q_s, $\bar{C}_s$ are density, specific attached (or separated) mass and velocity vector of the carried phase:

$$\rho_s = \sum_i \rho_{si} f_i \quad (28)$$

$$\bar{C}_s = \frac{1}{\rho_s}\sum_i \rho_{si} f_i \bar{C}_{si} \quad (29)$$

$$q_s = \sum_i q_{si} \quad (30)$$

and

$$\sum_i \chi_{si} = \sum_{i,j} \chi_{sji} = 0 \quad (31)$$

according to the law of physical and chemical transformations.

From (27) with $q_s = 0$, as a special case, we obtain:

$$\frac{\partial}{\partial t}(\rho_s \phi_s) + div(\rho_s \phi_s \bar{C}_s) = \chi \quad (32)$$

3. Let us determine the mass transfer equation for the multiphase mixture as a whole. It can be obtained by summing either the integral phase equations or their differential forms. By adding (20) and (27), and also taking into account the expression for the velocity vector of a multi-phase (two-phase) medium (Aslanov et al., 2022):

$$\bar{C} = \frac{\rho_s \phi_s \bar{C}_s + \rho_f \phi_f \bar{C}_f}{\rho_s \phi_s + \rho_f \phi_f} \quad (33)$$

we can write:

$$\frac{\partial \rho}{\partial t} + div(\rho \bar{C}) = q \quad (34)$$

where: ρ, q are the density and specific attached (or separated, then $q < 0$) mass of a two-phase (inhomogeneous) medium:

$$\rho = \rho_f \phi_f + \rho_s \phi_s \quad (35)$$

$$q = q_f + q_s \quad (36)$$

Considering the mass concentration of the carrier χ_f and carried χ_s phases:

$$\chi_f = \frac{(\rho_f \phi_f)}{\rho}, \quad \chi_s = \frac{(\rho_s \phi_s)}{\rho} \quad (37)$$

expression (33) can be represented as:

$$\bar{C} = \chi_f \bar{C}_f + \chi_s \bar{C}_s \quad (38)$$

Equation (38) will be referred to as the differential equation of mass transfer of a multicomponent medium. In the absence of an attached (or separated) mass of the mixture, equation (34) coincides with (Gusev et al., 2023):

$$\frac{\partial \rho}{\partial t} + div(\rho \bar{C}) = 0 \quad (39)$$

Let us write the mass transfer (continuity) equations for the phases and the multi-phase medium as a whole in the Cartesian coordinate system

$$\frac{\partial}{\partial t}(\rho_f \phi_f) + \frac{\partial}{\partial x}(\rho_f \phi_f u_f) + \frac{\partial}{\partial y}(\rho_f \phi_f v_f) + \frac{\partial}{\partial z}(\rho_f \phi_f w_f) = q_f + \chi \quad (40)$$

$$\frac{\partial}{\partial t}(\rho_s \phi_s) + \frac{\partial}{\partial x}(\rho_s \phi_s u_s) + \frac{\partial}{\partial y}(\rho_s \phi_s v_s) + \frac{\partial}{\partial z}(\rho_s \phi_s w_s) = q_s + \chi \quad (41)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = q \quad (42)$$

Using the numerical numbering of coordinates ($x = x_1$, $y = x_2$, $z = x_3$) and speed ($u = C_1$, $v = C_2$, $w = C_3$), equations (40)–(42) can be represented in a more compact form. For example, equation (42) will have the following form (Habibov et al., 2022):

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{3} \frac{\partial}{\partial x_k}(\rho C_k) = q \qquad (43)$$

Conclusions

Based on the results of scientific research, the authors recommend using the improved approach for deriving phenomenological equations proposed in this work when calculating the hydromechanics of multi-phase flows consisting of mass transfer equations.

References

Abbasov N.M., Zeinalov R.I., Azizova O.M., Imranova S.N., 2006. Dynamic models of heat exchangers. *Chemistry and Technology of Fuels and Oils this link is disabled*, 42(1): 25–29. DOI: 10.1007/s10553-006-0022-2.

Alder M.D., 2001. Introduction to Mathematical Modelling: A Menagerie of Difference Equations. *HeavenforBooks.Com.*, 1–209

Allaire G., Craig A.A., 2007. Numerical Analysis and Optimization: An Introduction to Mathematical Modeling and Numerical Simulation (Numerical Mathematics and Scientific Computation). *Oxford University Press*, 1–472.

Aslanov J.N., Huseynli Z.S., Abbasov N.M., Sharifova A.V., 2022. Durability study of specialized sealing elements. *International Journal on Technical and Physical Problems of Engineering*, 14(3): 8–13.

Gusev A.L., Jabbarov T.G., Mamedov Sh.G., Malikov R., Hajibalaev N.M., Abdullaeva S.D., Abbasov N.M., 2023. Production of hydrogen and carbon in the petrochemical industry by cracking of hydrocarbons in the process of heat utilization in steel production. *International Journal of Hydrogen Energy this link is disabled*, 48(49): 14954-14963. DOI: 10.1016/j.ijhydene.2022.12.341.

Habibov I.A., Sadigova T.Y., Abasova S.M., 2022. Assessment of the effect of ferromagnetic liquids on oil recovery of bituminous sands. *Eastern-European Journal of Enterprise Technologies*, 3(6(117): 47–52. DOI: 10.15587/1729-4061.2022.257104.

Kuznetsov V.L., 2017. Mathematical Modeling: Textbook. *MGTUGA*.

Mathematical modeling, 2014. Methodology and methods for developing mathematical models of mechanical systems and processes. Part I. Modeling of systems and processes. Third edition, revised and supplemented: Textbook. *MGTUGA*, 1–108. ISBN 5-86311-428-2.

Shen S.S.P., 2012. Introduction to modern mathematical modeling – Developing Mathematical Models Using Data. *Wiley & Sons, Inc*, 1–32.

Timofeeva A.S., 2014. Hydrodynamics of two-phase systems. *Stary Oskol, «Metallurgiya», TNT*, p.264

Umnov A.E., 2012. Methods of mathematical modeling: Textbook. *MIPT,* 295. ISBN 5-7417-0189-2.

Zaliznyak V.E., Zolotov O.A., 2023. Introduction to mathematical modeling: textbook for secondary vocational education. *Yurayt Publishing House, Moscow*, 1–133. ISBN 978-5-534--13307-3.

Zvonarev S.V., 2019. Fundamentals of mathematical modeling: Textbook. *Ural Publishing House, un-ta, Yekaterinburg*, 1–112. ISBN 978-5-7996-2576-4.

Gasim Amir MAMEDOV, Ph.D.
Associate Professor at the Department of Industrial Engineering
Azerbaijan State Oil and Industry University
16/21 Azadliq Ave, AZ1010, Baku, Azerbaijan
E-mail: *qasim11_5858@mail.ru*

Natiq Mayis ABBASOV, Ph.D.
Associate Professor at the Department of Industrial Engineering
Azerbaijan State Oil and Industry University
16/21 Azadliq Ave, AZ1010, Baku, Azerbaijan
E-mail: *natiq.abbasov.1966@mail.ru*

Rauf Khalil MALIKOV, Ph.D.
Associate Professor at the Department of Industrial Engineering
Azerbaijan State Oil and Industry University
16/21 Azadliq Ave, AZ1010, Baku, Azerbaijan
E-mail: *malikov.rauf@mail.ru*

Mahluqa Surxay RAHIMOVA, Ph.D.
Assistant Professor at the Department of Mechanics
Azerbaijan State Oil and Industry University
16/21 Azadliq Ave, AZ1010, Baku, Azerbaijan
E-mail: *rahimova_mahluqa@mail.ru*