Nafta-Gaz 2025, no. 2, pp. 109–116, DOI: 10.18668/NG.2025.02.03

Mathematical modeling of an elastoplastic problem for a fractured massif, underground workings

Modelowanie matematyczne zagadnienia sprężysto-plastycznego dla spękanego masywu w wyrobiskach podziemnych

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ABSTRACT: This article considers the problem of plane elastoplasticity of stress distribution in a rock mass weakened by a circular excavation. The components of stress, strain, and displacement in the elastic and inelastic field, as well as the dimensions and shape of the contour separating them, have been determined. The convergence of results between the analytical and numerical solutions of the problem is evaluated. The complexity of elastoplastic problems lies in the fact that the shape and dimensions of the plastic zone are not known in advance and must be determined. The main advances in solving plane elastoplastic problems for isotropic bodies with circular holes are related to the complete coverage of the hole by the plastic deformation zone. Currently, the corresponding mathematical problem for an ideally plastic body is often reduced to a boundary problem for a biharmonic equation in an unknown boundary region, which must be identified during the process of solving the elastoplastic problem. In the directions of the X and Y axes, external, uniformly distributed loads are applied at infinity. These loads can either be unequal $\lambda \neq 1$ or equal $\lambda = 1$ (where λ is the lateral thrust coefficient). The magnitude of these loads is such that a plastic deformation zone forms around the excavation, completely covering its contour. Deformation and failure of the rock mass occurs under specified strain conditions in the elastically compressed part of the massif. In both elastic and plastic regions, the hypothesis of medium continuity is preserved. Since movement of the rock mass along the longitudinal axis of the excavation is restricted, the case of plane deformation is considered. The solution of the problem requires determining the components of stress, strain, and displacements in the elastic and inelastic regions, as well as the size and shape of the contour L separating these regions. The most difficult case of the problem formulated above is the case when the external forces are unequal, that is, the lateral thrust coefficient λ is not equal to unity. In this case, the boundary between the plastic and elastic regions (contour L) takes the shape of an ellipse. The final expressions of this solution are highly complex, complicating their analysis and practical application. It is assumed that the stress level is such that the excavation is entirely covered by the plastic zone, and that there is a crack in the elastic zone of the massif. Perturbation methods and analytic function theory are employed to solve the problem.

Key words: stress-strain state, elastoplastic problem, zone of inelastic deformations, fractured rock mass, stress intensity factors.

STRESZCZENIE: Celem niniejszego artykułu jest analiza sprężysto-plastycznego rozkładu naprężeń w górotworze osłabionym wyrobiskiem o przekroju kołowym. Określono składowe naprężenia, odkształcenia i przemieszczenia w polu sprężystym i niesprężystym, a także wymiary i kształt oddzielającego je konturu. Dokonano oceny zbieżności wyników między analitycznymi i numerycznymi rozwiązaniami tego problemu. Złożoność problemu sprężysto-plastycznego wynika z faktu, że kształt i wymiary strefy plastycznej nie są z góry znane i muszą być zdefiniowane. Największy postęp w rozwiązywaniu problemów sprężysto-plastycznych dla ciał izotropowych z okrągłymi wyrobiskami (otworami) związany jest z uzyskaniem pełnego pokrycia otworu przez strefę odkształcenia plastycznego. Obecnie analogiczny problem matematyczny w przypadku idealnie plastycznego ciała jest często redukowany do zagadnienia brzegowego dla równania biharmonicznego w nieznanym obszarze brzegowym, który musi być zidentyfikowany podczas procesu rozwiązywania zagadnienia sprężysto-plastycznego. W kierunkach osi X i Y działają zewnętrzne, równomiernie rozłożone obciążenia na dużych odległościach. Obciążenia te mogą być nierównomierne $\lambda \neq 1$ lub równomierne $\lambda = 1$ (gdzie λ jest współczynnikiem naporu bocznego). Wielkość tych obciążeń jest na tyle duża, że wokół wyrobiska tworzy się strefa odkształceń plastycznych, całkowicie pokrywająca jego kontur L. Do odkształcenia i zniszczenia górotworu dochodzi w określonych warunkach odkształcenia w sprężysto ściskanej części masywu. Zarówno w strefie sprężystej, jak i plastycznej, zachowana jest hipoteza o ciągłości ośrodka. Ponieważ ruch górotworu wzdłuż osi podłużnej wyrobiska jest ograniczony, rozpatrywany jest przypadek deformacji płaskiej. Rozwiązanie problemu wymaga określenia składowych naprężeń, odkształceń i przemieszczeń w obszarze sprężystym i niesprężystym, a także wielkości i kształtu konturu L oddzielającego te obszary. Najtrudniejszym przypadkiem sformułowanego powyżej problemu jest przypadek, gdy siły zewnętrzne są nierówne, tzn. współczynnik parcia bocznego λ nie jest równy jedności. W takim przypadku granica pomiędzy

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Article contributed to the Editor: 04.04.2024. Approved for publication: 17.01.2025.

obszarem plastycznym i sprężystym (kontur L) przyjmuje kształt elipsy. Ostateczne wyrażenia tego rozwiązania są bardzo złożone, co komplikuje ich analizę i praktyczne zastosowanie. Zakłada się, że poziom naprężeń jest taki, że wyrobisko jest całkowicie pokryte strefą plastyczną oraz że w strefie sprężystej masywu występuje pęknięcie. Do rozwiązania problemu zastosowano metody perturbacyjne i teorię funkcji analitycznych.

Słowa kluczowe: stan naprężenie-odkształcenie, zagadnienie sprężysto-plastyczne, strefa odkształceń niesprężystych, spękany górotwór, współczynniki intensywności naprężeń.

Introduction

At sufficiently high values of external loads, areas of plastic deformation appear near excavations, pits, recesses, and other similar structural or technological connections. The consideration of plastic zones is particularly important for calculating the strength of structures and constructions. Sokolovsky (1969) obtained the solution to the elastoplastic problem for a plane with a hole under conditions of plasticity. It was shown that the contour of the interface between elastic and plastic deformations is very close to an ellipse. The stress function, which describes the stress in the plastic zone, is not biharmonic. Sokolovsky (1969) also provided an approximate solution to the elastoplastic problem for a massif weakened by two identical circular excavations under conditions of plasticity. It was assumed that the value of the stress in the massif and the distance between the workings were such that the circular developments were completely covered by the corresponding plastic zones and that those plastic zones did not intersect. The article provides a solution to the planar elastoplastic problem of stress distribution in rock mass weakened by circular excavation under the influence of tectonic and gravity forces. Practical experience with mine workings shows that crack initiation and destruction occur in the rock massif. Therefore, during the design stage of mine workings, it is necessary to take into account the possibility of crack formation and conduct a limit analysis of the fractured rock mass weakened by the crack.

In this context, solving the elastoplastic problem for a rock mass with a circular excavation, while considering the presence of cracks in the elastic zone during loading, is of great importance. To date, no such studies have been reported.

Problem statement

Let us consider the stress-strain state of a homogeneous, isotropic elastic rock mass near a circular, long, single horizontal excavation located at a depth H below the ground surface and not affected by mining operations. The radius of the excavation is R_0, and a uniformly distributed load of intensity P_0 is applied to its contour, equal to the resistance of the support. We assume that the rock medium, which has a compressive strength R_c, is weightless within the zone of influence of the excavation (Figure 1). The greater the depth of the excavation, the smaller the error resulting from this idealization. As Yerzhanov (1959) and Mikhlin (1934) showed, this error does not exceed 1%.

Mirsalimov (1987), based on Yerzhanov's work, concluded that within the upper layer of the lithosphere, where mining works are carried out, it is possible to consider carrying out a wide range of mining works in horizontally formed sedimentary rocks, taking into account the geological conditions. The stresses in the entire rock mass are distributed hydrostatically, that is, $\lambda = 1$. In this case, the solution of the problem is significantly simplified, because the contour of the ellipse L turns into a circle. The calculation scheme used to solve the problem is shown in Figure 1.

The most difficult situation of the formulated problem occurs when the external forces applied along the horizontal and vertical axes are unequal, that is, when the lateral thrust coefficient λ is not equal to one ($\lambda \neq 1$). The calculation scheme shown in Figure 1 is quite general, because when there are tangential stresses at infinity (for example, due to non-tectonic factors), it is always possible to choose the coordinate axes in the direction of the main stresses. As a result, the distribution of stresses at infinity will correspond to that assumed in the problem ($\sigma_x = \sigma_x^\infty$, $\sigma_y = \sigma_y^\infty$, $\tau_{xy} = 0$).

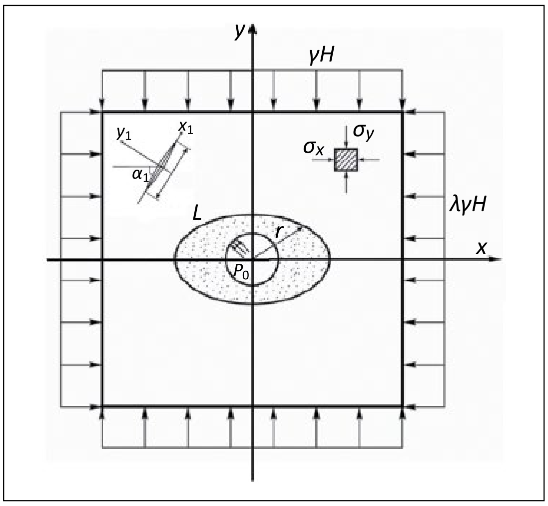

Figure 1. Calculation scheme of an elastoplastic problem for a fractured rock mass weakened by a circular excavation

Rysunek 1. Schemat obliczeniowy dla zagadnienia sprężysto-plastycznego dla spękanego górotworu osłabionego kolistym wykopem

When the mass is loaded around the circular excavation, stress concentration occurs. At sufficiently large values of external loads, σ_x^∞, σ_y^∞, P_0, a plastic zone is formed. It is assumed that the plastic zone completely surrounds the circular working area, and there is a rectilinear crack outside the plastic zone, that is, in the elastic zone (Figure 1).

In this case, it is assumed that there are no tangential stresses in the plastic region ($\tau_{r\theta} = 0$), and, as a result, the stress state is axisymmetric (Savash et al., 2019). Let us denote the stress components in the plastic region with the index 1 placed on top, and the stresses in the elastic region without the index. We assume that the weighty elastic half-space $y < H$ is weakened by one tunnel, which is a cylinder with an axis parallel to the surface of the half-space.

Then the boundary conditions are defined as follows:
on the development contour

$$\tau_{r\theta}^1\Big|_{R=R_0} = 0, \quad \sigma_r^1\Big|_{R=R_0} = p_0 \tag{1}$$

at infinity

$$\sigma_x^\infty = \lambda\gamma H, \quad \sigma_y^\infty = \gamma H, \quad \tau_{xy}^\infty = 0 \tag{2}$$

where:

σ_x^∞, σ_y^∞ – horizontal and vertical normal stresses, respectively; τ_{xy}^∞ – tangential stresses; $\lambda = \mu/(1-\mu)$ – coefficient of lateral thrust of the rock; μ – Poisson's ratio of the rock; γ – average density of the rock mass; $(H-\gamma)$ – depth of the considered point of the massif from the earth's surface.

At an arbitrary point in the rock mass with coordinates X, Y, the stress components satisfy the equilibrium equations

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0, \quad \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} = 0 \tag{3}$$

and the condition of compatibility of deformations:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \times (\sigma_x + \sigma_y) = 0 \tag{4}$$

In the region of plastic deformations, in addition, the physical equation takes the following form:

$$\sigma_\theta + \sigma_r = 2k\left(\frac{A}{r^2} - B\right) \tag{5}$$

Here and below, all quantities that have the dimension of length and displacement are related to the excavation radius R_0.

A strength reduction function $f(r)$ is introduced into the strength condition, which determines the law by which the strength of rocks for uniaxial compression or adhesion in the surrounding mine working changes depending on the relative radius $r(r = R/R_0$,where R_0 is the working radius and R is the current radius).

The principle for choosing an analytical expression for the strength reduction function is essentially the same. For example, in the "$\sigma - r$" coordinate system, experimental data are approximated by a monotonic curve, the ordinates of which increase from some value close to or equal to zero at the excavation contour to the strength of the untouched massif R_c at the interface between the plastic and elastic regions (Mehtiyev and Tanriverdiyev, 2023). To some degree, the known analytical expressions for the strength reduction function correspond to this principle. However, it is quite obvious that if, when constructing the initial physical model, the rock medium is assumed to be continuous, the form of the function $f(r)$ must correspond to this initial condition. In particular, in both the plastic and elastic regions, the stress function $F(r)$ must be biharmonic, and therefore it will have a single specific expression.

To find out the type of strength reduction function, we proceed as follows. Let us write the initial relations in the polar coordinate system. The equations of equilibrium and compatibility of deformations take the following forms (Timoshenko and Goodyear, 1975):

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{6}$$

$$\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = 0 \tag{7}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta^2}\right)\cdot(\sigma_\theta + \sigma_r) = 0 \tag{8}$$

where r, θ are polar coordinates.

We present the robustness condition in a fairly general form as follows:

$$(\sigma_\theta - \sigma_r)^2 + 4\tau_{r\theta}^2 = 4k^2 f^2(r) \tag{9}$$

where k is a constant depending on the initial physical prerequisites included in the strength condition.

Let us introduce the stress function in such a way that the following relations are satisfied in the plastic region:

$$\sigma_r = \frac{1}{r}\frac{dF}{dr}, \quad \sigma_\theta = \frac{d^2F}{dr^2}, \quad \tau_{r\theta} = 0 \tag{10}$$

It is obvious that in this form, the stress function always satisfies the equilibrium equations.

To determine the analytical expression for the strength reduction function, we substitute expressions (10) into (8) and (9). We get a system of equations:

$$\frac{1}{r}\frac{dF}{dr} - \frac{d^2F}{dr^2} = \pm 2kf(r) \tag{11}$$

$$\nabla\nabla F = 0 \tag{12}$$

where ∇ – Laplace operator.

Solving equation (11) by the constant variation method, we obtain the following expression for the stress function:

$$F(r) = kr^2 \int f(r) \cdot r^{-1} dr - k \int rf(r) dr + C_1 r^2 + C_2 \quad (13)$$

where C_1 and C_2 are arbitrary integration constants.

To determine the components of the stress field in the plastic region, we introduce the stress function $F(r)$, which is related to them by dependencies (10) and is determined in accordance with expression (11):

$$F(r) = \\ = 2k\left[r^2\left(C_1 + \frac{B}{4}\right) - \frac{B}{2}r^2 \ln r - \frac{A}{2}\left(\ln r + \frac{1}{2}\right)\right] + \quad (14) \\ + C_1 r^2 + C_2$$

Using the second boundary condition on the excavation contour (1), we find the values of the integration constants:

$$C_1 = \frac{P_0}{2k} + \frac{A}{4}, \quad C_2 = 0 \quad (15)$$

Then, taking into account (15), expression (14) will take the form:

$$F(r) = \\ = 2k\left[\frac{r^2}{2}\left(\frac{A}{2} + \frac{B}{2} + \frac{P_0}{k}\right) - \frac{B}{2}r^2 \ln r - \frac{A}{2}\left(\ln r + \frac{1}{2}\right)\right] \quad (16)$$

Using expression (16) and formula (15), we determine the stress components in the plastic region:

$$\sigma_r^{(1)} = \frac{1}{r} \cdot \frac{dF}{dr} = 2k\left[\frac{A}{2}\left(1 - \frac{1}{r^2}\right) - B \ln r + \frac{P_0}{2k}\right]$$
$$\sigma_\theta^{(1)} = \frac{d^2F}{dr^2} = 2k\left[\frac{A}{2}\left(1 + \frac{1}{r^2}\right) - B(\ln r + 1) + \frac{P_0}{2k}\right] \quad (17)$$
$$\tau_{r\theta}^{(1)} = 0$$

At the boundary L between the plastic and elastic regions, the stresses are continuous:

$$\sigma_x^1 = \sigma_x, \quad \sigma_y^1 = \sigma_y, \quad \tau_{xy}^1 = \tau_{xy} \quad (18)$$

The edges of a straight crack are considered free from external loads. At the center of the straight crack, we place the origin of the local coordinate system x_1Oy_1, with the x_1 axis coinciding with the crack line, and making an angle α_1 with the x axis (Figure 1).

To find the stressed state in the elastic zone of the massif, we use the following boundary conditions:

$\sigma_r - i\sigma_{r\theta} = \sigma_r^1 - i\sigma_{r\theta}^1$ along L,

$\sigma_{y_1} = 0, \quad \tau_{x_1 y_1} = 0$ at the edges of the crack.

To determine the unknown boundary L between the elastic and plastic regions, we use the following condition:

$$\sigma_\theta = \sigma_\theta^1 \quad (19)$$

The additional condition (19) allows us to find the required contour function.

Solution method

The problem posed concerns the unknown boundary L separating the elastic and plastic zones. We will search for the unknown boundary L as follows

$$r = \rho(\theta) = a_0 + \varepsilon H(\theta)$$

in which the function $\rho(\theta)$ is to be determined. Here $\varepsilon = R^0/a_0$ is a small parameter, and R^0 is greatest height of profile roughness L from circle $r = a_0$.

We assume that any A1 function can be expressed by a trigonometric Fourier series (Mirsalimov and Hasanov, 2022).

$$H(\theta) = \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta)$$

We search for the functions (stresses, displacements, stress intensity coefficients) in the form of the separation of the small parameter ε:

$$\sigma_r = \sigma_r^{(0)} + \varepsilon\sigma_r^{(1)} + \ldots; \quad \tau_{r\theta} = \tau_{r\theta}^{(0)} + \varepsilon\tau_{r\theta}^{(1)} + \ldots;$$
$$\sigma_\theta = \sigma_\theta^{(0)} + \varepsilon\sigma_\theta^{(1)} + \ldots; \quad (20)$$
$$u = u^{(0)} + \varepsilon u^{(1)} + \ldots; \quad \upsilon = \upsilon^{(0)} + \varepsilon\upsilon^{(1)} + \ldots$$

in which we neglect, for simplicity, terms containing ε to a power higher than the first.

Each approximation satisfies the system $\sigma_r^{(j)}, \sigma_\theta^{(j)}, \tau_{r\theta}^{(j)} u^{(j)}$, $v^{(j)} (j = 0,1,2,\ldots)$ of differential equations of the plane problem in elasticity theory. Next, we use the perturbation method. We obtain the values of the stress tensor components at $r = \rho(\theta)$ by expanding into a series the expressions for the stresses in the vicinity of $r = a_0$.

Using the well-known formulas (Muskhelishvili, 1977) for stress components σ_n and τ_{nt} (n, t are natural coordinates), we obtain the boundary conditions of the problem on contour $r = a_0$ in the form:

for zero approximation

$$\sigma_r^{(0)} = \sigma_r^1; \quad \tau_{r\theta}^{(0)} = \tau_{r\theta}^1, \text{ at } r = a_0 \quad (21)$$
$$\sigma_{y_1}^{(0)} = 0, \quad \tau_{x_1 y_1}^{(0)} = 0, \text{ at the edges of the crack}$$

for the first approximation

$$\sigma_r^{(1)} = N, \quad \tau_{r\theta}^{(1)} = T, \text{ at } r = a_0 \quad (22)$$
$$\sigma_{y_1}^{(1)} = 0, \quad \tau_{x_1 y_1}^{(1)} = 0, \text{ at the edges of the crack}$$

here:

$$N = \frac{2\tau_{r\theta}^{(0)}}{a_0} \frac{dH(\theta)}{d\theta} - H(\theta)\frac{\partial\sigma_r^{(0)}}{\partial r}, \text{ at } r = a_0 \quad (23)$$
$$T = (\sigma_\theta^{(0)} - \tau_{r\theta}^{(0)}) \frac{1}{a_0} \frac{dH(\theta)}{d\theta} - H(\theta)\frac{\partial\tau_{r\theta}^{(0)}}{\partial r}$$

The Kolosov-Muskhelishvili relations for the elastic zone are as follows (Muskhelishvili, 1966):

$$\sigma_x+\sigma_y=\sigma_r+\sigma_\theta=2\left[\Phi(z)+\overline{\Phi(z)}\right] \qquad (24)$$

$$\sigma_y-\sigma_x+2i\tau_{xy}=(\sigma_\theta-\sigma_r+2i\tau_{r\theta})e^{-2i\theta}=2\left[\bar{z}\Phi'(z)+\Psi(z)\right]$$

$$2\mu(u+i\upsilon)=2\mu e^{i\theta}(u_r+i\upsilon_\theta)=\chi\phi(z)-z\overline{\phi'(z)}-\overline{\Psi(z)}$$

$$\phi'(z)=\Phi(z),\quad \psi'(z)=\Psi(z),\quad (z=x+iy),\quad \chi=3-4\nu$$

In zero approximation, we write the boundary conditions of the problem (21) in the following form:

$$\Phi_0(z)+\overline{\Phi_0(z)}-e^{2i\theta}\left[\bar{z}\Phi_0'(z)+\Psi_0(z)\right]=\sigma_r^1-i\tau_{r\theta}^1,\ \text{at } z=a_0e^{i\theta} \qquad (25)$$

$$\Phi_0(t_1)+\overline{\Phi_0(t_1)}+x_1\overline{\Phi_0'(t_1)}+\overline{\Psi_0(t_1)}=0,\ \text{at the edges of the crack}$$

where:
$\Phi_0(z)$ and $\Psi_0(z)$ – complex potentials in zero approximation,
t_1 – affix of crack edge points in zero approximation.

In zero approximation, in the region occupied by the elastic material of the mountain rock, we look for the complex potentials $\Phi_0(z)$ and $\Psi_0(z)$ in the following form (Panasyuk et al., 1966):

$$\Phi_0(z)=\Phi_0^0(z)+\Phi_0^1(z),\quad \Psi_0(z)=\Psi_0^0(z)+\Psi_0^1(z) \qquad (26)$$

$$\Phi_0^0(z)=\sum_{k=0}^{\infty}a_k^0z^{-k},\quad \Psi_0^0(z)=\sum_{k=0}^{\infty}b_k^0z^{-k} \qquad (27)$$

$$\Phi_0^1(z)=\frac{1}{2\pi}\int_{-l_1}^{l_1}\frac{g_1^0(t)}{t-z}dt+$$

$$+\frac{1}{2\pi}\int_{-l_1}^{l_1}\left\{\left(-\frac{1}{z}-\frac{\overline{T}_1}{1-z\overline{T}_1}\right)e^{i\alpha_1}g_1^0(t)+\frac{1-T_1\overline{T}_1}{\overline{T}_1(1-z\overline{T}_1)^2}e^{-i\alpha_1}\overline{g_1^0(t)}\right\}dt \qquad (28)$$

$$\Psi_0^1(z)=\frac{1}{2\pi}e^{-2i\alpha_1}\int_{-l_1}^{l_1}\left[\frac{\overline{g_1^0(t)}}{t-z_1}-\frac{\overline{T}_1e^{i\alpha_1}}{(t-z_1)^2}g_1^0(t)\right]dt+$$

$$+\frac{1}{2\pi}\int_{-l_1}^{l_1}\left\{\left[\frac{1}{T_1z}-\frac{\overline{T}_1}{z(1-z\overline{T}_1)}+\frac{\overline{T}_1^2}{(1-z\overline{T}_1)^2}\right]e^{i\alpha_1}g_1^0(t)\right\}+$$

$$+\left[\frac{1}{z\overline{T}_1}+\frac{2(1-e^{-i\alpha_1})}{z^2\overline{T}_1^2}-\frac{z+T_1}{z(1-z\overline{T}_1)^2}-\frac{2\overline{T}_1(ze^{-i\alpha_1}-T_1)}{(1-z\overline{T}_1)^3}\right]e^{-i\alpha_1}g_1^0(t)dt$$

where $T_1=te^{i\alpha_1}+z_1^0$, $z_1=e^{-i\alpha_1}(z-z_1^0)$, $g_1^0(t_1)$ – any function that characterizes the opening of crack edges in zero approximation,

$$g_1^0(t_1)=\frac{2\mu}{2(1+\chi)}\frac{d}{dx}\left\{\left[(u_1^0)^++(u_1^0)^-\right]+i\left[(\upsilon_1^0)^++(\upsilon_1^0)^-\right]\right\}$$

$$a_0^0=\frac{1}{4}(\sigma_x^\infty+\sigma_y^\infty),\ b_0^0=\frac{1}{2}(\sigma_y^\infty-\sigma_x^\infty),\ a_1^0=\frac{A_1^0a_0}{1+k},\ b_1^0=\frac{kA_1^0a_0}{1+k},$$

$$a_2^0=\frac{1}{2}(\sigma_y^\infty-\sigma_x^\infty)a_0^2+\overline{A}_2^0a_0^2,\ b_2^0=\frac{1}{2}(\sigma_y^\infty-\sigma_x^\infty)a_0-\overline{A}_0^0a_0^2,$$

....

$$a_n^0=\overline{A}_n^0\,(n\geq 3),\ b_n^0=(n-1)a_0^2a_{n-2}^0-a_0^nA_{-n+2}^0\,(n\geq 3),$$

$$\sigma_r^1-i\tau_{r\theta}^1=\sum_{k=-\infty}^{\infty}A_k^0e^{ik\theta}$$

By applying the functions (26)–(28) to the boundary condition (25) at the crack edge in the elastic zone, we obtain the singular integral equation for the unknown function $g_1^0(x_1)$.

$$\int_{-l_1}^{l_1}\left[R_{11}(t,x)g_1^0(t)+S_{11}(t,x)\overline{g_1^0(t)}\right]dt=\pi F_0(x)\quad |x|\leq l_1 \qquad (29)$$

where $F_0(x)=-\left[\Phi_0^0(x)+\overline{\Phi_0^0(x)}+x\overline{\Phi_0'^0(x)}+\overline{\Psi_0^0(x)}\right]$, x, t, z_1^0 and $l_1\,a_0$ are dimensionless quantities associated with; R_{11}, S_{11} are determined using known formulas (see Panasyuk et al., 1976).

An additional condition should be added to equation (29).

$$\int_{-l_1}^{l_1}g_1^0(t)dt=0 \qquad (30)$$

This ensures unambiguous displacement when going around the contour of the internal crack in the elastic zone of the material.

The singular integral equation (29), with the additional condition (30), using the algebraization procedure (Panasyuk et al., 1976; Mirsalimov, 1987), is reduced to a system of M algebraic equations with respect to the approximate values $g_1^0(t_m)\,(m=1,2,...,M)$ of the desired function at the nodal points:

$$\frac{1}{M}\sum_{m=1}^{M}l_1\left[g_1^0(t_m)R_{11}(l_1t_m,l_1x_r^0)+\overline{g_1^0(t_m)}S_{11}(l_1t_m,l_1x_r^0)\right]=$$

$$=F_0(x_r^0) \qquad (31)$$

$$\sum_{m=1}^{M}g_1^0(t_m)=0,\ r=1,2,...,M-1$$

where:

$$t_m=\cos\frac{2m-1}{2M}\pi,\ (m=1,\,2,...,M),\ x_r^0=\cos\frac{\pi r}{M},$$
$$(r=1,\,2,...,M-1).$$

For stress intensity factors in the zero approximation, we find:
in the vicinity of the crack tip at $t_1=l_1$

$$K_I^{(0)l_1}-iK_{II}^{(0)l_1}=\sqrt{\pi l_1}\sum_{m=1}^{M}(-1)^mg_1^0(t_m)\operatorname{ctg}\frac{2m-1}{4M}\pi \qquad (32)$$

in the vicinity of the crack tip at $t_1=-l_1$

$$K_I^{(0)(-l_1)}-iK_{II}^{(0)(-l_1)}=\sqrt{\pi l_1}\sum_{m=1}^{M}(-1)^{m+M}g_1^0(t_m)\operatorname{tg}\frac{2m-1}{4M}\pi \qquad (33)$$

Using formulas (24) and (26), the stress components in the elastic zone are found in the zero approximation.

Knowing the stress state in the elastic zone in the zero approximation, we formally find the functions N and T according to relations (23).

The boundary conditions of the problem for determining the stress-strain state in the elastic zone in the first approximation (22), using complex Kolosov–Muskhelishvili potentials (24), will be written in the following form:

$$\Phi_1(z)+\overline{\Phi_1(z)}-e^{2i\theta}\left[\bar{z}\Phi_1'(z)+\Psi_1(z)\right]=N-iT \text{ at } z=a_0e^{i\theta} \quad (34)$$

$$\Phi_1(t_1)+\overline{\Phi_1(t_1)}+t_1\overline{\Phi_1'(t_1)}+\overline{\Psi_1(t_1)}=0 \text{ at the edges of the crack.}$$

The solution of the boundary value problem (34) is constructed similarly to the zero approximation:

$$\Phi_1(z)=\Phi_1^0(z)+\Phi_1^1(z), \quad \Psi_1(z)=\Psi_1^0(z)+\Psi_1^1(z) \quad (35)$$

where the complex potentials $\Phi_1(z)$, $\Psi_1(z)$ are determined by similar formulas (28), where the function $g_1^0(t)$ should be replaced by the function $g_1^1(t)$, and the analytical functions $\Phi_1^0(z)$ and $\Psi_1^0(z)$ are sought in the form of power series.

$$\Phi_1^0(z)=\sum_{k=0}^{\infty}a_k^1 z^{-k}, \quad \Psi_1^0(z)=\sum_{k=0}^{\infty}b_k^1 z^{-k} \quad (36)$$

For coefficients a_k^1 and b_k^1, we have:

$$a_0^1=0,\ b_0^1=0,\ a_1^1=\frac{\overline{A_1}a_0}{1+k},\ b_1^1=\frac{kA_1a_0}{1+k} \quad (37)$$

$$a_2^1=\overline{A_2}a_0^2,\ b_2^1=-\overline{A_0}a_0^2$$

....

$$a_n^0=\overline{A}_n^0 a_0^n \ (n\ge 2),\ b_n^1=(n-1)a_0^2a_{n-2}^1-a_0^nA_{-n+2}^0 \ (n\ge 3)$$

$$N-iT=\sum_{k=-\infty}^{\infty}A_ke^{ik\theta}$$

By satisfying the functions (35)–(36) to the boundary condition (34) at the edges of the cracks in the elastic zone, and after some mathematical transformations, we get the following singular integral equation for the function $g_1^1(t)$:

$$\int_{-l_1}^{l_1}\left[R_{11}(t,x)g_1^1(t)+S_{11}(t,x)\overline{g_1^1(t)}\right]dt=\pi F_1(x) \quad |x|\le l_1 \quad (38)$$

where $F_1(x)=-\left[\Phi_1^0(x)+\overline{\Phi_1^0(x)}+x\overline{\Phi_1'^0(x)}+\overline{\Psi_1^0(x)}\right]$, x,t,z_1^0 and l_1 are dimensionless values related to a_0; R_{11}, S_{11} are determined using known formulas.

It is necessary to add an additional condition to equation (38)

$$\int_{-l_1}^{l_1}g_1^1(t)dt=0 \quad (39)$$

ensuring unambiguous movements when going around the contour of a crack in the elastic zone to the first approximation.

Similar to the zero approximation, the integral equation (38) under the additional condition (39) is reduced to a system of M algebraic equations for approximate values $g_1^1(t_m)$, $(m=1,2,...,M)$ of the desired function at nodal points:

$$\frac{1}{M}\sum_{m=1}^{M}l_1\left[g_1^1(t_m)R_{11}(l_1t_m,l_1x_r)+\overline{g_1^1(t_m)}S_{11}(l_1t_m,l_1x_r)\right]=F_1(x_r) \quad (40)$$

$$\sum_{m=1}^{M}g_1^1(t_m)=0,\ r=1,2,...,M-1$$

where:

$$t_m=\cos\frac{2m-1}{2M}\pi,\ (m=1,2,...,M),$$

$$x_r=\cos\frac{\pi r}{M},\ (r=1,2,...,M-1).$$

For the stress intensity coefficients in the vicinity of the crack tips, we find:

$$K_I^{(1)l_1}-iK_{II}^{(1)l_1}=\sqrt{\pi l_1}\sum_{m=1}^{M}(-1)^m g_1^1(t_m)\operatorname{ctg}\frac{2m-1}{4M}\pi \quad (41)$$

$$K_I^{(1)(-l_1)}-iK_{II}^{(1)(-l_1)}=\sqrt{\pi l_1}\sum_{m=1}^{M}(-1)^{m+M} g_1^1(t_m)\operatorname{tg}\frac{2m-1}{4M}\pi \quad (42)$$

It should be noted that in subsequent approximations, the solution of the problem for the elastic zone is constructed in a similar way.

The resulting system of equations (37), (40) is not yet closed. These equations include the unknown coefficients a_k and b_k of the expansion of the desired function $H(\theta)$, which describes the contour L of the interface between elastic and plastic deformations. To determine the coefficients a_k, b_k and to establish the missing equations, it is necessary to determine the normal circumferential stress σ_θ on contour L. We find the circumferential stress σ_θ on contour L using formulas (20), (24), (26), (35).

Based on the solution obtained, we find σ_θ up to first-order values relative to the small parameter in the following form:

$$\sigma_\theta^e=\sigma_\theta^0\Big|_{r=a_0}+\varepsilon\left[H(\theta)\frac{\partial\sigma_\theta^{(0)}}{\partial r}+\sigma_\theta^{(1)}\right]\Bigg|_{r=a_0} \quad (43)$$

Therefore, the function $H(\theta)$ must be defined in such a way that the additional condition (19) is satisfied on the contour L.

To construct the missing equations that allow us to find the coefficients a_0, a_k, b_k of the expansion of the function $\rho(\theta)$, we use the principle of least squares. The circumferential stress σ_θ on the contour L is a function of the independent variable polar angle θ and $(2m+1)$ parameters a_0, a_k, b_k $(k=1, 2, ..., m)$.

The unknown parameters a_0, a_k, b_k are constant and must be determined. To find them, we carry out a series of calculations. We divide the segment $[0, 2\pi]$ of the change in the variable θ into M_1 parts, where $M_1>2m+1$:

$$\theta_j=\theta_1+j\Delta\theta,\ \Delta\theta=\frac{2\pi}{M_1},\ (j=1,2,...,M_1) \quad (44)$$

We calculate the circumferential normal stress at the splitting points:

$$\sigma_\theta^e(\theta_j)=F(\theta_j,a_0,a_k,b_k),\ (j=1,2,...,M_1) \quad (45)$$

Thus, it is required to find such values of the unknown parameters a_0, a_k, b_k that will best provide the values of the circumferential normal stress function σ_θ on L with σ_θ^1 values.

The principle of least squares states that the most likely values of the parameters will be those for which the sum of squared deviations will be the smallest:

$$U = \sum_{j=1}^{M_1} F\left[(\theta_j, a_0, a_k, b_k) - \sigma_\theta^p(\theta_j)\right]^2 \to \min.$$

Using the necessary condition for the extremum of a function U of several variables, we obtain $(2m + 1)$ equations with $(2m + 1)$ unknowns:

$$\frac{\partial U}{\partial a_0} = 0,\ \frac{\partial U}{\partial a_k} = 0,\ \frac{\partial U}{\partial b_k} = 0,\ (k = 1, 2, ..., m) \qquad (46)$$

The system of equations (46) closes the previously obtained systems (37), (40) of the problem.

A joint numerical solution of the resulting systems of algebraic equations makes it possible to find approximate values of the function $g_1(t_m)$ ($m = 1, 2, ..., M$), stress intensity coefficients in the vicinity of the crack tips and coefficients a_0, a_k, b_k ($k = 1, 2, ..., m$).

In real materials, in the vicinity of the crack tip there is always a zone in which plastic deformations occur. Various approaches are used to evaluate these plastic deformations. In the case of quasi-brittle fracture for such materials (for example, various rocks and low-plasticity metals), the well-known Irvine–Orowan concept of quasi-brittle fracture is used. In this case, the plastic zone at the crack tip is taken into account by the Irwin correction. Irwin showed that the presence of ductility causes a crack to behave as if its length is longer than it actually is. The size of the plastic zone and the intensity of plastic deformations in it are entirely controlled by the stress intensity factor and the properties of the material. In this case, due to its smallness, it is recommended (or possible) not to introduce a plastic correction for plane deformation.

Numerical example

For the numerical calculation, the case of a crack in the elastic zone of the massif:

$$\alpha_1 = 30^0;\ \frac{l_1}{a_0} = 0{,}07;\ z_1^0 = 1{,}2a_0 e^{\frac{i\pi}{12}}$$

was chosen. The calculation was performed using the Gaussian method with the choice of the principal element. $M = 25$, $M_1 = 82$ were assumed. The results of calculations for the coefficients a_0, a_k, b_k of the expansion of the function $\rho(\theta)$ describing the elastoplastic boundary are given in Table 1, calculated for specific values of external loads at $R = 2$.

After solving the resulting algebraic systems, the stress intensity factors in the vicinity of the crack tips $x_1 = \pm l_1$ in the elastic zone were found using the formulas:

at $t_1 = l_1$

$$K_I - iK_{II} = \sqrt{\pi l_1} \sum_{m=1}^{M} (-1)^m \left[g_1^0(t_m) + \varepsilon g_1^1(t_m)\right] ctg \frac{2m-1}{4M}\pi \quad (47)$$

at $t_1 = -l_1$

$$K_I - iK_{II} = \sqrt{\pi l_1} \sum_{m=1}^{M} (-1)^{m+M} \left[g_1^0(t_m) + \varepsilon g_1^1(t_m)\right] \mathrm{tg} \frac{2m-1}{4M}\pi \quad (48)$$

Figure 2 shows plots of the stress intensity factors K_I and K_{II} as a function of crack length l_1/a_0.

By changing the values of the parameters α_1 and z_1^0, which characterize the position of the crack, it is possible to study various cases of the location of a straight crack in the elastic zone and its influence on the interface between elastic and plastic deformations and on the parameters of rock mass destruction.

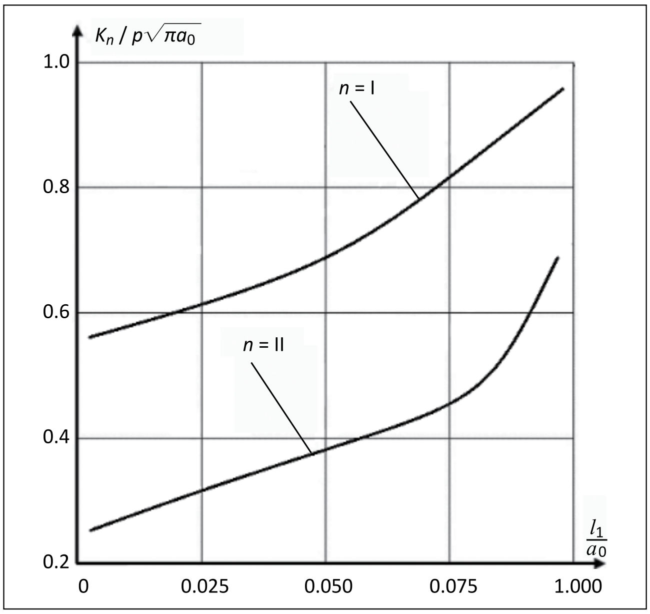

Figure 2. Dependence of stress intensity factors on crack length

Rysunek 2. Zależność współczynników intensywności naprężeń od długości pęknięcia

Table 1. Fourier coefficients of the elastoplastic boundary

Tabela 1. Współczynniki Fouriera granicy sprężysto-plastycznej

N	σ_0/k	p/k	C	σ_x^∞/k	σ_y^∞/k	a_0	a_1	a_2	b_1	b_2	b_3
1	0.05			–1.12	–1.30	1.037	–0.344	–1.05	–1.57	1.074	0.640
2		–0.8	0.972	–1.18	–1.45	1.092	–0.078	–1.06	–1.093	1.058	0.089

Conclusions

An approximate method for solving the elastoplastic problem is proposed for the case when there is a straight crack in the elastic zone of the rock mass. A closed system of algebraic equations has been derived, the solution of which makes it possible to study the stressed state of the massif with full coverage of the mine workings by the plastic zone and in the presence of a crack in the elastic zone.

Given the mechanical and geometric characteristics of the rock mass, the obtained basic solution equations allow the use of numerical calculations to determine stress intensity factors. This facilitates the prediction of crack growth, as well as the determination of permissible level of defects in the rock mass and the limit values of external loads.

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